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A NOTE ON BREDA-ROBERTSON'S CONJECTURE

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Abstract

The continuous deformation of any spherical isometric folding into the standard spherical folding, f_s, defined by $f_s(x, y, z) = (x, y, |z|)$, remains an open problem since 1989. We show that this conjecture is restricted to the class of primitive foldings and it is exhibited a spherical folding within this class, where the difficulty of deformation is evidenced.

1. Introduction

The theory of isometric foldings was introduced in 1977 by Robertson [4]. It emerged as a formulation, in the language of Riemannian geometry, of the physical action of crumpling a sheet of paper and then crushing it flat against a desk top. For related work, see also [5].

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The intrinsic geometry of crumpling-crushing process can be modelled mathematically by regarding both the paper and the desk top as two dimensional flat Riemannian manifolds M and N, respectively, and by representing the process itself as a map, $f : M \to N$, which sends piecewise geodesic segments to piecewise geodesic segments of the same length, Figure 1. The same definition applies for the general situation, where M and N are Riemannian manifolds of any dimension.

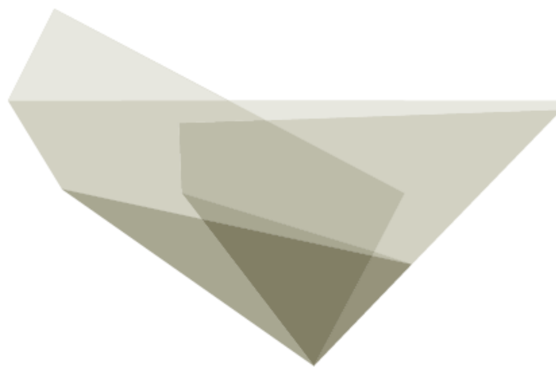

Figure 1. A planar isometric folding.

This class of maps can be formally defined as follows.

Considering a smooth (C^{∞}) Riemannian manifold M, a map $\alpha : I = [a, b] \to M$ is a *zig-zag* on M, if there exists a subdivision, $a = a_0 < a_1 < \cdots < a_r = b$ of I, such that for all $j = 1, \ldots, r$, the restriction $\alpha_j = \alpha|_{[a_{j-1}, a_j]}$ is a geodesic segment on M parameterized by arc-length. Thus, the length of α is $L(\alpha) = \sum_{j=1}^{r} L(\alpha_j) = b - a$.

Definition 1.1. Let M and N be smooth connected Riemannian manifolds of finite dimensions. A map $f : M \to N$ is said to be an *isometric folding of M into N* if, for every zig-zag $\alpha : I \to M$, the induced path $\alpha_* = f \circ \alpha : I \to N$ is a zig-zag on N. It follows that $L(\alpha) = L(\alpha_*)$.

Our interest is focused in the set of isometric foldings of the Riemannian sphere, $\mathcal{F}(S^2)$.

Definition 1.2. Let $f, g \in \mathcal{F}(S^2)$. We shall say that f is deformable into g, if there exists a map, (homotopy) $H : [0, 1] \times S^2 \to S^2$, such that

(i) H is continuous;

(ii) for each $t \in [0, 1]$, H_t defined by $H_t(x) = H(t, x)$, $x \in S^2$, is an isometric folding;

(iii) $H(0, x) = f(x)$ and $H(1, x) = g(x)$, $\forall x \in S^2$.

Considering the compact open topology on $\mathcal{F}(S^2)$, then f is deformable in g iff they belong to the same path connected component.

A point $x \in S^2$, where the isometric folding $f : S^2 \to S^2$ fails to be differentiable is called a *singularity* of f. The set of all singularities of f is denoted by $\sum f$. An isometric folding f is said to be *non trivial* if $\sum f \neq \emptyset$, i.e., f is not an isometry of S^2.

The *standard spherical isometric folding*, denoted by f_s, is defined by

$$f_s(x, y, z) = (x, y, |z|), \quad \forall (x, y, z) \in S^2.$$

A general description of $\sum f$, for any $f \in \mathcal{F}(S^2)$, was given by Robertson in [4]. This description can be stated as follows: For each $x \in \sum f$, the singularities of f near x form the image of an even number of geodesic rays emanating from x and making alternated angles $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n$, where

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = \pi. \tag{1.1}$$

In other words, the singularity set of an isometric folding in S^2 can be seen as an embedded graph of even valency satisfying the angle folding relation (1.1). In Figure 2 is illustrated a singularity set near a vertex of valency six.

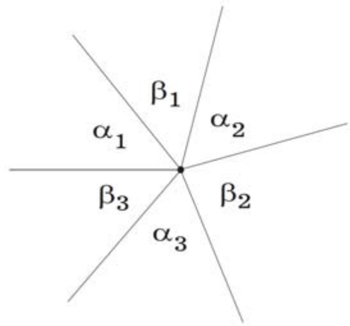

Figure 2. The angle folding relation (with $n = 3$).

The compactness of the sphere assures that the singularity set of any spherical isometric folding (as an embedded graph of S^2) is connected with finitely many regions.

Definition 1.3. By a spherical folding tiling (*f*-tiling, for short), we mean an edge-to-edge finite polygonal-tiling τ of S^2, whose underlying graph is of the type described in (1.1).

We shall denote by $\mathcal{T}(S^2)$, the set of all *f*-tilings of S^2, identifying the singularity sets of non-trivial foldings with spherical *f*-tilings.

In [1], it was established that any non-trivial isometric folding (with Hopf degree zero) in the Euclidian plane is deformable into the *standard planar folding* $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $f(x, y) = (x, |y|)$ and was conjectured that

$$\textit{any non-trivial spherical isometric folding is deformable into } f_s. \tag{1.2}$$

This statement is known as the Breda-Robertson's conjecture. It should be pointed out that a proof given for the sphere can not be a simple adaptation from the one used for the plane, since here the dilatations had played a crucial role.

2. Deformation on $\mathcal{F}(S^2)$

Let f and $g \in \mathcal{F}(S^2)$. The set of singularities, $\sum f$, determines, up to an isometry, the associated spherical isometric folding, f, as seen in [3]. In other words, $\sum f = \sum g$ iff there exists a spherical isometry i, such that $g = i \circ f$. On the other hand, a deformation H of f into g induces a deformation on its sets of singularities and reciprocally.

Proposition 2.1. *Let $f, g \in \mathcal{F}(S^2)$. If f and g are deformable into f_s, then $f \circ g$ is deformable into f_s.*

Proof. The result follows observing that if H_1 is a deformation of f into f_s and H_2 is a deformation of g into f_s, then $H : [0, 1] \times S^2 \to S^2$, defined by $H(t, x) = H_2(t, H_1(t, x))$ is a deformation of $f \circ g$ into $f_s \circ f_s = f_s$. □

The classification of spherical dihedral f-tilings started in 2004. It was proved that, there are exactly five f-tilings, namely, $R^3, U_1, U_2, U_3, U_4 \in \mathcal{T}(S^2)$ (with the notation used in [2]), with prototiles an isosceles spherical triangle T of angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$ and a spherical rhombus Q with opposite pairs of angles $(\frac{2\pi}{3}, \frac{\pi}{2})$, see Figure 3.

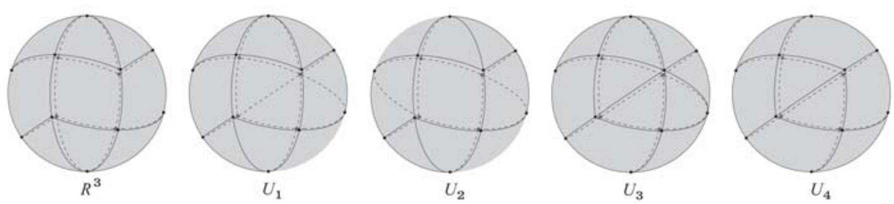

Figure 3. The dihedral f-tilings of the sphere with prototiles Q and T.

An isometric folding $f \in \mathcal{F}(S^2)$ is said to be *simple*, if $\sum f$ is a great circle of S^2. In [3], it was proved that if f is of the form $f = f_1 \circ \cdots \circ f_n$, where each $f_i (i = 1, \ldots, n)$ is a simple isometric folding, then f is deformable into f_s.

Proposition 2.2. *The isometric foldings associated to the spherical tilings* $R^3, U_1, U_3,$ *and* U_4 *verify the Breda-Robertson's conjecture* (1.2).

Proof. Let $\phi, \psi_1, \psi_3,$ and ψ_4 be the spherical isometric foldings with set of singularities the f-tilings $R^3, U_1, U_3,$ and U_4, respectively.

We show that ϕ and ψ_i, $i = 1, 3, 4,$ are finite compositions of simple foldings. It should be noted that if $f = g \circ h$, for some non-trivial isometric foldings g and h, then $\sum f = \sum h \cup h^{-1}(\sum g)$.

In Figure 4, we present a geometrical sketch to prove that ϕ is composition of five simple foldings, namely, $f_1, f_2, f_3, f_4,$ and f_5. We illustrate step by step the spherical f-tiling correspondent to the set of singularities of f_1, $f_2 \circ f_1$, $f_3 \circ f_2 \circ f_1$, $f_4 \circ f_3 \circ f_2 \circ f_1$, and $\phi = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$.

The arrows indicate the way we should move vertices and edges allowing each folding to be joined to the standard one.

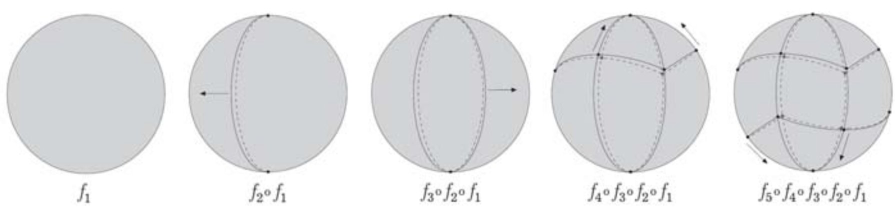

Figure 4. The singularity set $\sum \phi = R^3$, with $\phi = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$.

The isometric folding ψ_1 can also be seen as a composition of five simple foldings, namely, $f_1^1, f_2^1, f_3^1, f_4^1,$ and f_5^1. The set of singularities is illustrated step by step in Figure 5. As before, a geometrical sketch through arrows could be presented in order to illustrate the deformation.

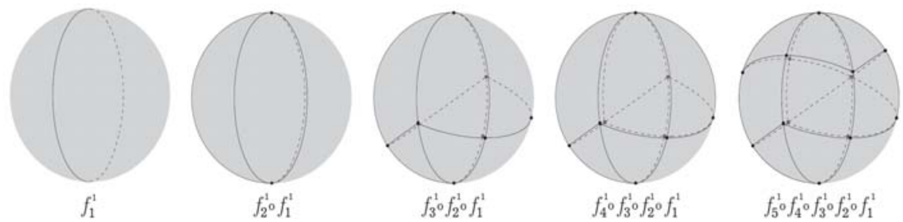

Figure 5. The singularity set $\Sigma \psi_1 = U_1$, with $\psi_1 = f_5^1 \circ f_4^1 \circ f_3^1 \circ f_2^1 \circ f_1^1$.

A similar procedure helps us to show that $\psi_3 = f_5^3 \circ f_4^3 \circ f_3^3 \circ f_2^3 \circ f_1^3$, for some simple foldings f_1^3, f_2^3, f_3^3, f_4^3, and f_5^3, see Figure 6.

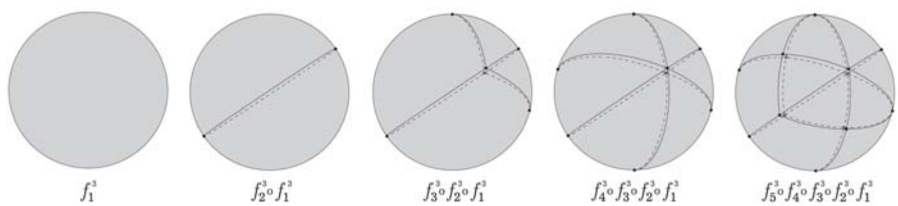

Figure 6. The singularity set $\Sigma \psi_3 = U_3$, with $\psi_3 = f_5^3 \circ f_4^3 \circ f_3^3 \circ f_2^3 \circ f_1^3$.

Finally, we prove that ψ_4 is composition of the simple foldings f_1^4, f_2^4, f_3^4, and f_4^4. The singularity set is exhibited in Figure 7.

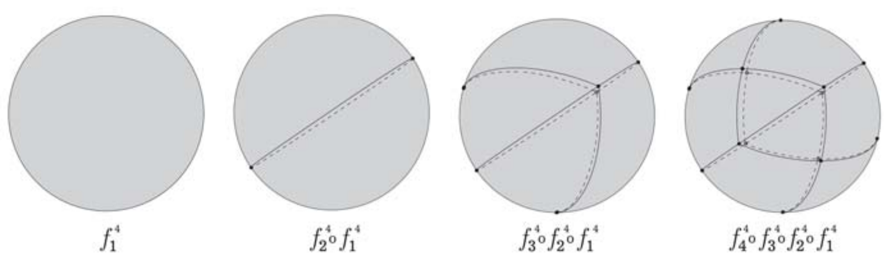

Figure 7. The singularity set $\Sigma \psi_4 = U_4$, with $\psi_4 = f_4^4 \circ f_3^4 \circ f_2^4 \circ f_1^4$.

$\square$

Definition 2.1. A non-trivial isometric folding $f \in \mathcal{F}(S^2)$ is said to be *primitive*, if there are no g, h non-trivial distinct spherical isometric foldings, such that $f = g \circ h$. Otherwise, f is called *non-primitive*.

By Proposition 2.1, the deformation stated in (1.2) is solved if any primitive folding is deformable into f_s.

Proposition 2.3. *If* $f \in \mathcal{F}(S^2)$ *is non-primitive, then* $\sum f$ *contains a proper sub-f-tiling. In other words, there is a subgraph* Γ *of* $\sum f$ *such that* $\emptyset \neq \Gamma \underset{\neq}{\subset} \sum f$ *and* $\sum f \setminus \Gamma$ *is an f-tiling.*

Proof. Suppose that $f = g \circ h$, for some non-trivial distinct isometric foldings g and h. As $\sum f = \sum h \cup h^{-1}(\sum g)$ and considering $\Gamma = h^{-1}(\sum g)$, we conclude that $\sum f \setminus \Gamma = \sum h$ is an f-tiling. □

Remark 1. If $f = f_n \circ \cdots \circ f_1$, for some $f_i \in \mathcal{F}(S^2)$, then

$$\Sigma(f_n \circ \cdots \circ f_2 \circ f_1) = \underbrace{\sum f_1}_{\Sigma_1} \cup \underbrace{f_1^{-1}(\sum f_2)}_{\Sigma_2} \cup \underbrace{(f_2 \circ f_1)^{-1}(\sum f_3)}_{\Sigma_3}$$

$$\cup \ldots \cup \underbrace{(f_{n-1} \circ \cdots \circ f_1)^{-1}(\sum f_n)}_{\Sigma_n}.$$

Denoting by

$$\Sigma_1 = \sum f_1 \quad \text{and} \quad \Sigma_k = (f_{k-1} \circ \cdots \circ f_1)^{-1}(\sum f_k), \quad k = 2, \ldots, n,$$

one gets the sub-f-tilings

$$\tau_k = \bigcup_{i=1}^{k} \Sigma_i, \quad k = 1, \ldots, n.$$

The reciprocal of Proposition 2.3 is false. In Figure 8-I is illustrated the set of singularities of a primitive spherical isometric folding containing a sub-f-tiling, as shown in Figure 8-II.

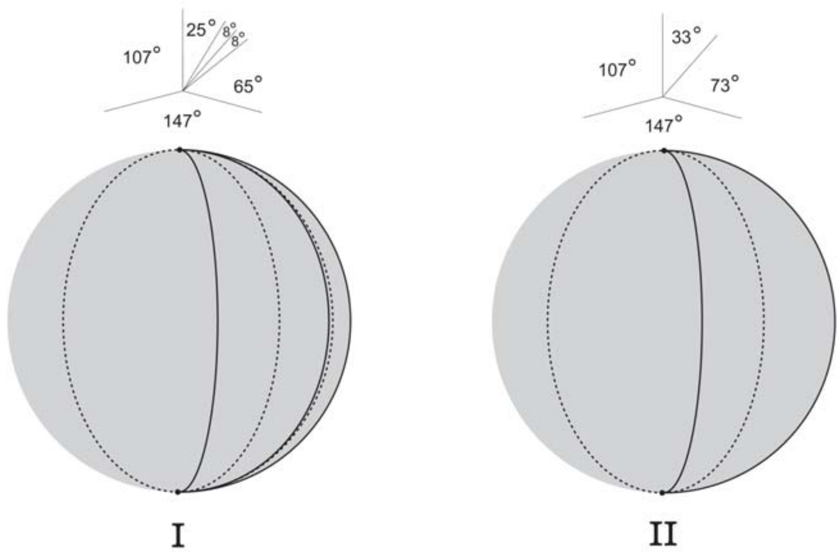

Figure 8. A primitive isometric folding with a sub-*f*-tiling.

Consider the *f*-tiling $U_2 \in \mathcal{T}(S^2)$ with prototiles Q and T (Figure 3) and let e be an edge of U_2. Then, a subgraph $\Gamma \underset{\neq}{\subset} U_2$ containing e, such that $U_2 \setminus \Gamma$ is an *f*-tiling, does not exist. And so, by Proposition 2.3, any isometric folding with set of singularities U_2 is primitive.

It follows that if $f \in \mathcal{F}(S^2)$, such that $\sum f = U_2$, then there is no deformation of f into f_s, in the sense of moving vertices and edges as done in Proposition 2.2. In fact, any perturbation on vertices break the angle folding relation (1.1).

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